Chapter 3: 3.1 – 3.10 Practice TEST #2 Name: Solutions pd.: 3, 7

Mrs. Robertson Show work and circle answers date: ____________

1. Given: $f(x) = \sin(3x)$

   a. Find the derivative with respect to x.

   $f'(x) = \cos(3x) \cdot 3$

   $\boxed{f'(x) = 3\cos(3x)}$

   b. Find the slope of the line tangent to the function at $\frac{\pi}{3}$.

   $f'(\pi/3) = 3\cos(3 \cdot \pi/3)$

   $f'(\pi/3) = 3\cos(\pi)$

   $f'(\pi/3) = 3(-1)$

   $\boxed{f'(\pi/3) = -3}$

   c. Write the equation of the line tangent to the function at $\frac{\pi}{3}$. point $(\pi/3, 0)$

   $f(\pi/3) = \sin(3 \cdot \pi/3)$

   $= \sin(\pi)$

   $= 0$

   $y - 0 = -3(x - \pi/3)$

   $\boxed{y = -3x + \pi}$

2. If the position function (in feet per second) is given as $f(x) = -16x^2 + 59x + 22$, find the following information.

   a. What is the position of the object after 3 seconds? $f(3) = -16(3)^2 + 59(3) + 22 =$ 55 feet

   b. What is the velocity function? $f'(x) = -32x + 59$

   c. What is the instantaneous velocity at 2 seconds? $f'(2) = -32(2) + 59 =$ $\boxed{-5 \text{ ft/sec}}$

   d. What is the acceleration function? $f''(x) = -32$

   e. What is the instantaneous acceleration at 2 seconds? $f''(2) =$ $\boxed{-32 \text{ ft/sec}^2}$

3. Find $f'(x)$ if $f(x) = sin^2(\sqrt[3]{3x^2})$ $\left(\sin\left(\sqrt[3]{3x^2}\right)\right)^2$ $\frac{d}{dx}\left((3x^2)^{1/3}\right) = \frac{1}{3}(3x^2)^{-2/3}(6x)$

$$f'(x) = 2\left(\sin\left(\sqrt[3]{3x^2}\right)\right)\left(\cos\left(\sqrt[3]{3x^2}\right)\right)\left(\frac{1}{3}(3x^2)^{-2/3}(6x)\right)$$

$$f'(x) = \frac{12x\left(\sin\left(\sqrt[3]{3x^2}\right)\cos\left(\sqrt[3]{3x^2}\right)\right)}{3(3x^2)^{2/3}} = \boxed{\frac{4x\sin\left(\sqrt[3]{3x^2}\right)\cos\left(\sqrt[3]{3x^2}\right)}{(3x^2)^{2/3}}}$$

4. Find the derivative using the Chain Rule. $y = [(5x + 2)^2 + 6x]^3$

$$y' = 3[(5x+2)^2 + 6x]^2[2(5x+2)(5) + 6]$$

$$y' = 3[(5x+2)^2 + 6x]^2[10(5x+2) + 6]$$

$$y' = 3[(5x+2)^2 + 6x]^2[50x + 26]$$

$$\boxed{y' = [150x + 78][(5x+2)^2 + 6x]^2}$$

5. Find $\frac{d^2y}{dx^2}$ of $y = x^2\sin(x)$

$\boxed{y' = x^2\cos(x) + 2x\sin(x)}$ 1st derivative

$$y'' = -x^2\sin(x) + \underbrace{2x\cos(x) + 2x\cos(x)} + 2\sin(x)$$

$\boxed{y'' = -x^2\sin(x) + 4x\cos(x) + 2\sin(x)}$ 2nd derivative

6. Differentiate **implicitly** with respect to x. $y = 2xy^2$

$$\frac{d}{dx}[y = 2xy^2]$$

$$y' = (2x)(2yy') + 2y^2$$

$$y' = 4xyy' + 2y^2$$

$$y' - 4xyy' = 2y^2$$

$$y'(1 - 4xy) = 2y^2$$

$$\boxed{y' = \frac{2y^2}{1 - 4xy}}$$

7. Given: $y(x^3+2x)=5$ $\quad y=\frac{5}{x^3+2x}$

a.) Differentiate **implicitly** with respect to x.

$$y(3x^2+2)+y'(x^3+2x)=0$$

$$y'(x^3+2x)=-y(3x^2+2)$$

$$y'=\frac{-y(3x^2+2)}{x^3+2x}$$

$$y'=\frac{-5(3x^2+2)}{(x^3+2x)^2}$$

Same

b.) Solve for y and differentiate **explicitly**.

$$y=\frac{5}{x^3+2x}$$

$$y'=\frac{(x^3+2x)(0)-5(3x^2+2)}{(x^3+2x)^2}$$

$$y'=\frac{-5(3x^2+2)}{(x^3+2x)^2}$$

8. Find $\frac{dy}{dx}[tan^{-1}(x^2y)=x+xy]$ $\quad \frac{dy}{dx}\left[\tan^{-1}(x)\right]=\frac{1}{x^2+1}$

$$\frac{dy}{dx}\left[\tan^{-1}(x^2y)=x+xy\right]$$

$$\left(\frac{1}{(x^2y)^2+1}\right)\left(x^2y'+2xy\right)=1+xy'+y$$

$$\frac{x^2y'+2xy}{x^4y^2+1}=1+xy'+y$$

$$x^2y'+2xy=(1+xy'+y)(x^4y^2+1)$$

$$x^2y'+2xy=x^4y^2+1+xy'(x^4y^2+1)+x^4y^3+y$$

$$x^2y'+2xy=x^4y^2+1+x^5y^2y'+xy'+x^4y^3+y$$

$$x^2y'-x^5y^2y'-xy'=x^4y^2+1+x^4y^3+y-2xy$$

$$y'(x^2-x^5y^2-x)=x^4y^2+1+x^4y^3+y-2xy$$

$$\boxed{y'=\frac{x^4y^2+1+x^4y^3+y-2xy}{x^2-x^5y-x}}$$

9. Find the derivative of $f(x)=5x^3(e^{2x}-3)+\frac{3x}{x^2+2}$

$$f'(x)=5x^3(2e^{2x})+15x^2(e^{2x}-3)+\frac{(x^2+2)(3)-3x(2x)}{(x^2+2)^2}$$

$$f'(x)=10x^3e^{2x}+15x^2e^{2x}-45x^2+\frac{3x^2+6-6x^2}{(x^2+2)^2}$$

$$\boxed{f'(x)=10x^3e^{2x}+15x^2e^{2x}-45x^2+\frac{6-3x^2}{(x^2+2)^2}}$$

10. Prove the following derivative rule using trigonometric identities. $\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$

$$\frac{d}{dx}[\sec(x)] = \frac{d}{dx}\left[\frac{1}{\cos(x)}\right]$$

$$= \frac{\cos(x)(0) - 1(-\sin(x))}{\cos^2(x)}$$

$$= \frac{\sin(x)}{\cos^2(x)}$$

$$= \frac{1}{\cos(x)} \cdot \frac{\sin(x)}{\cos(x)}$$

$$= \boxed{\sec(x)\tan(x)}$$

11. Differentiate implicitly then find the equation of the tangent at ~~(3,2)~~ change to $(1,0)$.

$2x^2 + xy - y^2 = 2$

$$4x + xy' + y - 2yy' = 0$$

$$xy' - 2yy' = -4x - y$$

$$\left(y' = \frac{-4x - y}{x - 2y}\right)$$

$$y' = \frac{-4(1) - 0}{1 - 2(0)} = -4$$

$$y - 0 = -4(x - 1)$$

$$\boxed{\cancel{y = 2x + 2}} \rightarrow \boxed{y = -4x + 4}$$

12. Find the derivative of $y = \sin^{-1}(x^2 - 1)$ if $\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}$

$$y' = \frac{1}{\sqrt{1-(x^2-1)^2}} \cdot 2x$$

$$\boxed{y' = \frac{2x}{\sqrt{1-(x^2-1)^2}}}$$

13. Differentiate: $f(x) = \ln(sin^2 x)$ $\quad \sin^2(x) = (\sin(x))^2$

$$f'(x) = \frac{1}{\sin^2(x)} \cdot 2\sin(x)\cos(x)$$

$$f'(x) = \frac{2\cos(x)}{\sin(x)} = \boxed{2\cot(x)}$$

14. Differentiate: $y = x^{x^2}$

$\ln y = \ln x^{x^2}$

$\frac{d}{dx}\left[\ln y = x^2 \ln(x)\right]$

$\frac{y'}{y} = x^2\left(\frac{1}{x}\right) + 2x\ln(x)$

$y' = y(x + 2x\ln(x))$

$\boxed{y' = x^{x^2}(x + 2x\ln(x))}$

15. A particle moves with position function $s = t^4 - 4t^3 - 20t^2 + 20t$ where $t \geq 0$.

(a) At what time does the particle have a velocity of 20 feet/second?

$s' = v = 4t^3 - 12t^2 - 40t + 20 = 20$

$4t^3 - 12t^2 - 40t = 0$

$4t(t^2 - 3t - 10) = 0$

$4t(t-5)(t+2) = 0$

$4t = 0 \quad t - 5 = 0 \quad t + 2 = 0$

$t = 0 \quad t = 5 \quad \cancel{t = -2}$

$\boxed{0 \text{ secs and } 5 \text{ seconds}}$

(b) At what time is the acceleration 0?

$s'' = v' = a = 12t^2 - 24t - 40 = 0$

$4(3t^2 - 8t - 10) = 0$

(not factorable)

$t = \frac{8 \pm \sqrt{(-8)^2 - 4(3)(-10)}}{2(3)}$

$t = \frac{8 \pm \sqrt{64 + 120}}{6} = \boxed{\frac{4 \pm \sqrt{46}}{3} \approx 3.59 \text{ secs}}$

16. The half-life of Radium is 1600 years. Suppose we have a 100 gram sample.

(a) Write an equation for the mass that remains after t years. Do not round r. Use the exact value of r.

$50 = 100e^{1600r}$

$.5 = e^{1600r}$

$\ln(.5) = 1600r$

$\boxed{\frac{\ln(.5)}{1600} = r}$

$\boxed{A(t) = 100e^{\frac{\ln(.5)}{1600}t} \quad \text{or} \quad A(t) = 100 \cdot 2^{\frac{-t}{1600}}}$

(b) How long will it take for 100 grams to decay to 90 grams? Round to the nearest year.

$90 = 100 \cdot 2^{-t/1600}$

$.9 = 2^{-t/1600}$

$\ln(.9) = \frac{-t}{1600}\ln(2)$

$\frac{\ln(.9)}{\ln(2)}(-1600) = t$

$t = 243.2049...$

$\boxed{t \approx 243 \text{ years}}$

17. How long (in years) will it take an investment to triple in value if the annual interest rate is 5.25% and the interest is compounded continuously?

$A = Pe^{rt}$

$3 = 1e^{.0525(t)}$

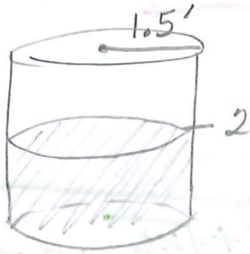
$\ln(3) = .0525t$

$\dfrac{\ln(3)}{.0525} = t$

$t = 20.925948...$

$\boxed{t \approx 21 \text{ years}}$

18. A water heater in the shape of a cylindrical tank is being filled with water at the rate of 1 ft^3/min. How fast is the height of the water in the tank rising when the height of the water is 2 feet? The radius of the tank is a constant 1.5 feet and the volume of a cylinder is given by: $V = \pi r^2 h$



given: $V' = 1\ ft^3/min$

find h'

$V = \pi (1.5)^2 h$

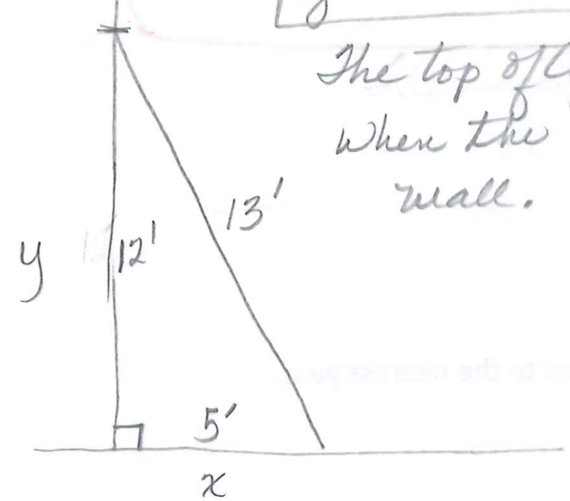
$V = 2.25\pi h$

$V' = 2.25\pi h'$

$1 = 2.25\pi h'$

$\boxed{\frac{4}{9\pi} ft/min} \Leftarrow$ OR $\frac{1}{2.25\pi} ft/min = h' \approx .14147\ ft/min$

19. A ladder that is **13** feet long rests against a vertical wall. If the ladder slides away from the wall at a rate of **2** feet per second, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is **5** feet from the wall?



$\boxed{\text{given } x' = 2\ ft/sec,\ \text{find } y'}$

The top of the ladder is 12' above the ground (floor) when the bottom of the ladder is 5' from the wall. ($13^2 = 5^2 + y^2$

$169 - 25 = y^2$

$\sqrt{144} = \sqrt{y^2}$

$y = 12$)

... To find y' we take the derivative of

$\frac{d}{dt}\left[x^2 + y^2 = 169\right]$

$2xx' + 2yy' = 0$

$2yy' = -2xx'$

$y' = -\frac{x x'}{y}$

$y' = \frac{-5}{12}(2)$

$y' = -\frac{10}{12}$

$\boxed{y' = -\frac{5}{6} ft/sec}$

20. Find the linearization of $f(x) = x^2 - 3x - 4$ when $a = 1$. $L(x) = f(a) + f'(a)(x-a)$

$f'(x) = 2x - 3 \quad f'(1) = -1 \quad f(1) = -6$

$L(x) = -6 + -1(x-1)$

$L(x) = -6 - (x-1) = -6 - x + 1$

$\boxed{L(x) = -5 - x}$

21. Given that $f(x) = \sqrt{x-1}$, $x = 5$, $\Delta x = 0.1$ answer the following questions.

(a) Find the differential. point

$f(5) = \sqrt{5-1} = 2 \quad (5, 2) \qquad dy = f'(x)\,dx \qquad \Delta x = dx = 0.1$

$f'(x) = \frac{1}{2}(x-1)^{-1/2} \qquad f'(5) = \frac{1}{2\sqrt{5-1}}$ a) $\boxed{dy = \left(\frac{1}{2\sqrt{x-1}}\right)dx}$

$f'(x) = \frac{1}{2\sqrt{x-1}} \qquad f'(5) = \frac{1}{4}$

(b) Find Δy and dy and compare. Write a statement for your comparison. $\Delta x = .1$

$dy = \left(\frac{1}{2\sqrt{5-1}}\right)(.1) \qquad \Delta y = f(x + \Delta x) - f(x)$

$dy = \frac{1}{4} \cdot \frac{1}{10} \qquad \Delta y = f(5.1) - f(5)$

$dy = \frac{1}{40} = .025 \qquad = \sqrt{5.1 - 1} - 2$

$= \sqrt{4.1} - 2$

$= 2.024845673 - 2$

$\Delta y = .024845673$

Δy rounded to the nearest 1000^{th} = dy

$.025 = .025$